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ON THE DIOPHANTINE EQUATION $G_n(x) = G_m(P(x))$: HIGHER-ORDER RECURRENCES

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Dedicated to Wolfgang M. Schmidt on the occasion of his 70th birthday.

Abstract. Let $\mathbf{K}$ be a field of characteristic 0 and let $(G_n(x))_{n=0}^{\infty}$ be a linear recurring sequence of degree d in $\mathbf{K}[x]$ defined by the initial terms $G_0, \ldots, G_{d-1} \in \mathbf{K}[x]$ and by the difference equation

$$G_{n+d}(x) = A_{d-1}(x)G_{n+d-1}(x) + \cdots + A_0(x)G_n(x), \quad \text{for } n \geq 0,$$

with $A_0, \ldots, A_{d-1} \in \mathbf{K}[x]$. Finally, let $P(x)$ be an element of $\mathbf{K}[x]$. In this paper we are giving fairly general conditions depending only on $G_0, \ldots, G_{d-1}$, on P, and on $A_0, \ldots, A_{d-1}$ under which the Diophantine equation

$$G_n(x) = G_m(P(x))$$

has only finitely many solutions $(n, m) \in \mathbb{Z}^2, n, m \geq 0$. Moreover, we are giving an upper bound for the number of solutions, which depends only on d. This paper is a continuation of the work of the authors on this equation in the case of second-order linear recurring sequences.

1. Introduction

Let $\mathbf{K}$ denote a field of characteristic 0. Without loss of generality we may assume that this field is algebraically closed. Let $A_0, \ldots, A_{d-1}, G_0, \ldots, G_{d-1} \in \mathbf{K}[x]$ and let the sequence of polynomials $(G_n(x))_{n=0}^{\infty}$ be defined by the d-th order linear recurring sequence

$$G_{n+d}(x) = A_{d-1}(x)G_{n+d-1}(x) + \cdots + A_0(x)G_n(x), \quad \text{for } n \geq 0. \tag{1.1}$$

Let

$$Q(T) = T^d - A_{d-1}(x)T^{d-1} - \cdots - A_0(x) \in \mathbf{K}[x][T]$$

denote the characteristic polynomial of the sequence $(G_n(x))_{n=0}^{\infty}$ and let $D(x)$ be the discriminant of $Q(T)$. It is clear that $D(x) \in \mathbf{K}[x]$. Moreover, let $\alpha_1(x), \ldots, \alpha_d(x)$ denote the roots of the characteristic polynomial $Q(T)$ in the splitting field $K(x)$ of $Q(T)$. The field $K(x)$ is a finite extension of $\mathbf{K}(x)$ of degree at most $d!$.

Assuming that $Q(T)$ has no multiple roots, i.e., $D(x) \neq 0$, it is well known that $(G_n(x))_{n=0}^{\infty}$ has a nice "analytic" representation. More precisely, there exist

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elements $g_1(x), \dots, g_d(x) \in K(x)$ such that

$$G_n(x) = g_1(x)\alpha_1(x)^n + \cdots + g_d(x)\alpha_d(x)^n \tag{1.2}$$

holds for all $n \geq 0$.

$(G_n(x))_{n=0}^{\infty}$ is called nondegenerate, if no quotient $\alpha_i(x)/\alpha_j(x), 1 \leq i < j \leq d$ is equal to a root of unity and it is called degenerate otherwise.

Many diophantine equations involving the recurrence $(G_n(x))_{n=0}^{\infty}$ were studied previously. For example, let us consider the equation

$$G_n(x) = s(x), \tag{1.3}$$

where $s(x) \in \mathbf{K}[x]$ is given. We denote by $N(s(x))$ the number of integers n for which (1.3) holds. From the theorem of Skolem, Mahler and Lech [13] it follows that $N(s(x))$ is finite for every $s(x)$ provided that the sequence is nondegenerate and that also $\alpha_1(x), \dots, \alpha_d(x)$ are not equal to a root of unity. Evertse, Schlickewei and Schmidt [9] proved that

$$N(s(x)) \leq e^{(6d)^{3d}} \tag{1.4}$$

under the same conditions as before. This is a direct consequence of the Main Theorem on S-unit equations over fields of characteristic 0, which we will state later on.

We mention that for $d = 2$, Schlickewei [17] had previously established an absolute bound for $N(s(x))$. His bound was substantially improved by Beukers and Schlickewei [3] who showed that $N(s(x)) \leq 61$. Very recently, Schmidt [18] obtained the remarkable result that for arbitrary nondegenerate complex recurrence sequences of order d one has $N(a) \leq C(d)$, where $a \in \mathbb{C}$ and $C(d)$ depends only (and in fact triply exponentially) on d.

Recently, the authors used new developments on S-unit equations over fields of characteristic 0 due to Evertse, Schlickewei and Schmidt (cf. [9]) to handle the equation $G_n(x) = G_m(P(x))$ for second-order linear recurring sequences $(G_n(x))_{n=0}^{\infty}$. Our result was: Let $p, q, G_0, G_1, P \in \mathbf{K}[x]$, $\deg P \geq 1$ and $(G_n(x))_{n=0}^{\infty}$ be defined by the second-order linear recurrence

$$G_{n+2}(x) = p(x)G_{n+1}(x) + q(x)G_n(x), \quad n \geq 0.$$

Assume that the following conditions are satisfied: $2\deg p > \deg q \geq 0$ and

$$\begin{aligned} \deg G_1 &> \deg G_0 + \deg p \geq 0, \quad \text{or} \\ \deg G_1 &< \deg G_0 + \deg q - \deg p. \end{aligned}$$

Then there are at most $e^{10^{18}}$ pairs of integers (n, m) with $n, m \geq 0, n \neq m$ such that

$$G_n(x) = G_m(P(x))$$

holds. We showed a second result in our paper: Let $\Delta(x) = p(x)^2 + 4q(x)$. Assume that

(1) $\deg \Delta \neq 0$,
(2) $\deg P \geq 2$,
(3) $\gcd(p, q) = 1$ and
(4) $\gcd(2G_1 - G_0 p, \Delta) = 1$.

Then there are at most $e^{10^{18}}$ pairs of integers (n, m) with $n, m \geq 0$ such that

$$G_n(x) = G_m(P(x))$$

holds.

The motivation for this equation was the following observation, which shows that the problem is nontrivial: Consider the Chebyshev polynomials of the first kind, which are defined by

$$T_n(x) = \cos(n \arccos x).$$

It is well known that they satisfy the following second-order recurring relation:

$$T_0(x) = 1, \quad T_1(x) = x,$$
$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x).$$

It is also well known and in fact easy to prove that

$$T_{2n}(x) = T_n(2x^2 - 1).$$

This example shows that some further conditions are needed.

By using function field analogs of S-unit equations, we were also able to give an upper bound for the cardinality of the set

$$\{(n, m) \in \mathbb{N} \mid n \neq m, \exists c \in \mathbf{K}^* \text{ such that } G_n(x) = c\, G_m(P(x))\}.$$

(Here c may vary with n, m.) Under the same assumptions as above we showed: The number of pairs of integers (n, m) with $n, m \geq 0, n \neq m$ for which there exists $c \in \mathbf{K}^*$ with

$$G_n(x) = c\, G_m(P(x))$$

is at most

$$C(p, q, P) = 10^{28} \cdot \log(C_1 \max\{2 \deg p, \deg q\}) \cdot (4e)^{8C_1 \deg q} \cdot 7^{4C_1 \deg q},$$

where $C_1 = 2(\deg P + 1)$.

The first author gave suitable extensions of the above results for third-order linear recurring sequences (cf. [10], [11]). He proved: Let $a, b, c, G_0, G_1, G_2, P \in \mathbf{K}[x]$, $\deg P \geq 1$ and $(G_n(x))_{n=0}^{\infty}$ be defined by the third-order linear recurring sequence

$$G_{n+3}(x) = a(x)G_{n+2}(x) + b(x)G_{n+1}(x) + c(x)G_n(x), \quad \text{for } n \geq 0. \tag{1.5}$$

Assume that the following conditions are satisfied: $3 \deg a > \deg c \geq 0, 2 \deg a > \deg b$ and $\deg a + \deg c > 2 \deg b$. Moreover, assume

$$\begin{aligned} \deg G_2 &> \deg G_1 + \deg a \geq 0, \quad \text{and} \\ \deg G_1 &> \deg G_0 + \tfrac{1}{2}(\deg c - \deg a). \end{aligned}$$

Then there are at most $e^{10^{24}}$ pairs of integers (n, m) with $n, m \geq 0, n \neq m$ such that

$$G_n(x) = G_m(P(x))$$

holds.

Moreover, we have: Let $a, b, c, G_0, G_1, G_2, P \in \mathbf{K}[x]$ and $(G_n(x))_{n=0}^{\infty}$ be defined by (1.5). Assume that

(1) $\deg D \neq 0, \deg q \neq 0$,
(2) $\deg P \geq 2$,
(3) $\gcd(c, D) = 1, \gcd(p, q) = 1$,

(4) $\gcd(G_2-\frac{2}{3}aG_1-\frac{2}{9}a^2G_0-bG_0,q)=1$, $\gcd(G_2^2-\frac{4}{3}bG_2G_0-\frac{1}{3}bG_1^2+\frac{4}{9}b^2G_0^2,D)=1$, and

(5) $\gcd(a,27c^2-4b^3)>1$,

where p,q are the coefficients of the characteristic polynomial of (1.5) in reduced form and D is the discriminant. Then there are at most $e^{10^{24}}$ pairs of integers (n,m) with $n,m\geq 0$ such that

$$G_n(x)=G_m(P(x))$$

holds.

It is the aim of this paper to present extensions of the results for linear recurrences of arbitrary order.

2. General results

To establish our first main result we need some preparations. By considering the initial terms of the recurrence we obtain the system of linear equations

$$G_j(x)=g_1(x)\alpha_1(x)^j+\cdots+g_d(x)\alpha_d(x)^j,\quad j=0,\dots,d-1 \tag{2.1}$$

for the algebraic functions $g_1(x),\dots,g_d(x)$. Let $\Delta(x)$ denote the determinant of this system. Then $\Delta(x)=\prod_{1\leq i<j\leq d}(\alpha_j(x)-\alpha_i(x))$; hence $D(x)=\Delta(x)^2$.

Define $\vec{A}=(A_0,\dots,A_{d-1}),\vec{G}=(G_0,\dots,G_{d-1})$ and $\vec{\alpha}_j=(1,\alpha_j,\dots,\alpha_j^{d-1})^T$, $j=1,\dots,d$. Applying Cramer's rule for the system of equations (2.1) we obtain

$$\Delta(x)g_1(x)=\det(\vec{G}^T(x),\vec{\alpha}_2(x),\dots,\vec{\alpha}_d(x)).$$

It is easy to see by induction that

$$\det(\vec{G}^T,\vec{\alpha}_2,\dots,\vec{\alpha}_d)=\left(\sum_{i=0}^{d-1}(-1)^{d-1-i}G_{d-1-i}S_i(\alpha_2,\dots,\alpha_d)\right)\prod_{2\leq i<j\leq d}(\alpha_j-\alpha_i),$$

where $S_i(\alpha_2,\dots,\alpha_d)$, $i=0,\dots,d-1$ denotes the i-th elementary symmetrical polynomial. Using Vieta's formulae we obtain

$$g_1(x)\alpha_1(x)\prod_{i=2}^{d}(\alpha_i(x)-\alpha_1(x))=\sum_{i=0}^{d-1}L_i(\vec{A},\vec{G})\alpha_1^i(x), \tag{2.2}$$

with some polynomial $L_i(\vec{A},\vec{G})\in\mathbb{Q}[\vec{A},\vec{G}]$, $i=0,\dots,d-1$. Since (2.1) is symmetrical in $\alpha_1(x),\dots,\alpha_d(x)$, the same relation holds if we replace the index 1 with another index $1\leq j\leq d$.

Now let

$$R=R_d(\vec{A},\vec{G})=\prod_{j=1}^{d}\left(\sum_{i=0}^{d-1}L_i(\vec{A},\vec{G})\alpha_j^i\right).$$

By the theorem on symmetrical polynomials $R(\vec{A},\vec{G})\in\mathbb{Q}[\vec{A},\vec{G}]$. To have some impression how complicated R is, we computed it for $d=3$:

$$\begin{aligned}R_3(\vec{A},\vec{G}) \quad = \quad & -G_2^3+(-A_1G_0+2A_2G_1)G_2^2\\ &+((A_1-A_2^2)G_1^2+(-3A_0+A_1A_2)G_0G_1-A_0G_0^2A_2)G_2\\ &+(-A_0-A_1A_2)G_1^3+(A_0A_2+A_1^2)G_0G_1^2-2A_0G_1A_1G_0^2\\ &+A_0^2G_0^3.\end{aligned}$$

Now we are in the position to state our first main result, which is a suitable analogue of the theorems in [12] for the number of solutions of

$$G_n(x) = c\, G_m(P(x)), \tag{2.3}$$

where $c \in \mathbf{K}^* = \mathbf{K}\backslash\{0\}$ is variable, for a linear recurring sequence $(G_n(x))_{n=0}^{\infty}$ of arbitrary large order.

Theorem 2.1. *Assume that the d-th order ($d \geq 2$) linear recurring sequence $(G_n(x))_{n=0}^{\infty}$ and the polynomial $P \in \mathbf{K}[x]$ satisfy the following conditions:*

- (i) *None of the roots and the quotients of distinct roots of the characteristic polynomial of $(G_n(x))_{n=0}^{\infty}$ is an element of $\mathbf{K}^*$,*
- (ii) $\deg P \geq 2$ *and* $\deg D \geq 1$,
- (iii) $\gcd(D, A_0) = 1$, *and*
- (iv) $\gcd(D, R(\vec{A}, \vec{G})) = 1$.

Then equation (2.3) *has at most*

$$C(d, A_0, D, P) := e^{(6d)^{4d}} \left(\log\left(d^{2d^2} \deg D(\deg P + 1)\right)\right)^{2d^2} (2ed)^{30d^3 d!^2 \deg A_0 \deg P}$$

solutions $(n, m) \in \mathbb{Z}^2$ with $n, m \geq 0, n \neq m$.

Remark 2.2. Observe that the conditions in Theorem 2.1 are suitable generalizations of the conditions of Theorem 3 in [12]. Moreover, observe that the structure of the bound is similar to that in [12], especially the dependence on $\deg D$ and $\deg P$.

It is also possible to get the conclusions from above for other types of assumptions.

Theorem 2.3. *Assume that the d-th order ($d \geq 2$) linear recurring sequence $(G_n(x))_{n=0}^{\infty}$ and the polynomial $P \in \mathbf{K}[x]$ satisfy the following conditions:*

- (i) *None of the roots and the quotients of distinct roots of the characteristic polynomial of $(G_n(x))_{n=0}^{\infty}$ is an element of $\mathbf{K}^*$,*
- (ii) $\deg P \geq 1$, *and* $\deg D \geq 1$,
- (iii) $\deg A_0 \geq 1$, $R(\vec{A}, \vec{G}) \neq 0$, *and*
- (iv) *the set of zeroes of A_0 is not equal to that of $A_0(P)$.*

Then equation (2.3) *has at most $C(d, A_0, D, P)$ solutions $(n, m) \in \mathbb{Z}^2$ with $n, m \geq 0, n \neq m$.*

The following proposition characterizes those polynomials A_0, P for which condition (iv) of the last theorem does not hold.

Proposition 2.4. *Let A_0 and P be nonconstant elements in $\mathbf{K}[x]$. Assume that A_0 and $A_0(P)$ have the same roots and let k be the number of different roots of A_0. Then there exist $a, b, c \in \mathbf{K}, a, c \neq 0$ such that if $k = 1$, then*

$$A_0(x) = a(x-b)^{\deg A_0} \quad \textit{and} \quad P(x) = c(x-b)^{\deg P} + b;$$

if $k \geq 2$, then either $P(x) = x$ or $P(x) = ax + b, a \neq 1$ and in this case,

$$A_0(x) = c\left(x + \frac{b}{a-1}\right)^s \prod_{i=1}^{r} \prod_{j=0}^{\ell-1} \left(x - a^j x_i - b\frac{a^j - 1}{a-1}\right),$$

where $x_1, \ldots, x_r \in \mathbf{K}$ are all different and a is a root of unity of order ℓ.

For the special case of the equation

$$G_n(x) = G_m(P(x)) \tag{2.4}$$

we can even show more.

Theorem 2.5. *Assume that the d-th order ($d \geq 2$) linear recurring sequence $(G_n(x))_{n=0}^{\infty}$ and the polynomial $P \in \mathbf{K}[x]$ satisfy the following conditions:*

(i) *None of the roots and the quotients of distinct roots of the characteristic polynomial of $(G_n(x))_{n=0}^{\infty}$ is a root of unity,*
(ii) $\deg P \geq 2$ *and* $\deg D \geq 1$,
(iii) $\gcd(D, A_0) = 1$, *and*
(iv) $\gcd(D, R(\vec{A}, \vec{G})) = 1$.

Then equation (2.4) *has at most*

$$e^{(12d)^{6d}}$$

solutions $(n, m) \in \mathbb{Z}^2$ with $n, m \geq 0, n \neq m$.

Remark 2.6. Observe that we can prove an upper bound for the number of solutions of (2.4) which does only depend on d.

Moreover, as an analogue of Theorem 2.3, we get:

Theorem 2.7. *Assume that the d-th order ($d \geq 2$) linear recurring sequence $(G_n(x))_{n=0}^{\infty}$ and the polynomial $P \in \mathbf{K}[x]$ satisfy the following conditions:*

(i) *None of the roots and the quotients of distinct roots of the characteristic polynomial of $(G_n(x))_{n=0}^{\infty}$ is a root of unity,*
(ii) $\deg P \geq 1$, *and* $\deg D \geq 1$,
(iii) $\deg A_0 \geq 1$, $R(\vec{A}, \vec{G}) \neq 0$, *and*
(iv) *the set of zeroes of A_0 is not equal to that of $A_0(P)$.*

Then equation (2.4) *has at most*

$$e^{(12d)^{6d}}$$

solutions $(n, m) \in \mathbb{Z}^2$ with $n, m \geq 0, n \neq m$.

Finally, we study a special instance of the above problem. Let $(G_n(x))_{n=0}^{\infty}$ be defined by (1.1) and let the initial polynomials be given by

$$G_0(x) = \cdots = G_{d-2}(x) = 0 \quad \text{and} \quad G_{d-1}(x) = 1.$$

Then we have

$$G_n(x) = \sum_{i=1}^{d} \frac{\alpha_i^n(x)}{Q'(\alpha_i(x))},$$

where

$$Q(T) = T^d - A_{d-1}(x)T^{d-1} - \cdots - A_0(x)$$

denotes the characteristic polynomial and $'$ means differentiation with respect to T. Observe that the discriminant $D(x)$ in this case is given by

$$D(x) = \prod_{i=1}^{d} Q'(\alpha_i(x)) = \prod_{j=1}^{d} \prod_{\substack{i=1 \\ i \neq j}}^{d} (\alpha_i(x) - \alpha_j(x)).$$

Applying Theorem 2.1 we get the following consequence:

Corollary 2.8. *Let $(G_n(x))_{n=0}^{\infty}$ be defined as above. Assume that $(G_n(x))_{n=0}^{\infty}$ and the polynomial $P \in \mathbf{K}[x]$ satisfy the following conditions:*

(i) *None of the roots and the quotients of distinct roots of the characteristic polynomial of $(G_n(x))_{n=0}^{\infty}$ is an element of $\mathbf{K}^*$,*
(ii) $\deg P \geq 2$ *and* $\deg D \geq 1$*, and*
(iii) $\gcd(D, A_0) = 1$.

Then we have:

(1) *Equation (2.3) has at most*

$$C(d, A_0, D, P) = e^{(6d)^{4d}} \left(\log\left(d^{2d^2} \deg D(\deg P + 1)\right)\right)^{2d^2} (2ed)^{30d^3 d!^2 \deg A_0 \deg P}$$

solutions $(n, m) \in \mathbb{Z}^2$ with $n, m \geq 0, n \neq m$.

(2) *Equation (2.4) has at most*

$$\min\{e^{(12d)^{6d}}, C(d, A_0, D, P)\}$$

solutions $(n, m) \in \mathbb{Z}^2$ with $n, m \geq 0, n \neq m$.

Observe that also Theorems 2.3 and 2.7 can be applied to this situation (but without any simplification of the assumption in general).

3. Auxiliary results

In this section we collect some important theorems which we will need in our proofs.

Let $\mathbf{K}$ be an algebraically closed field of characteristic 0, $n \geq 1$ an integer, $\alpha_1, \ldots, \alpha_n$ elements of $\mathbf{K}^*$ and Γ a finitely generated multiplicative subgroup of $\mathbf{K}^*$. A solution $(x_1, \ldots, x_n)$ of the so-called *weighted unit equation*

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 1 \text{ in } x_1, \ldots, x_n \in \Gamma \tag{3.1}$$

is called *nondegenerate* if

$$\sum_{j \in J} \alpha_j x_j \neq 0 \text{ for each nonempty subset } J \text{ of } \{1, \ldots, n\} \tag{3.2}$$

and *degenerate* otherwise. It is clear that if Γ is infinite and if (3.1) has a degenerate solution, then (3.1) has infinitely many degenerate solutions. For nondegenerate solutions we have the following result, which is due to Evertse, Schlickewei and Schmidt [9].

Theorem 3.1 (Evertse, Schlickewei and Schmidt)**.** *Let $\mathbf{K}$ be a field of characteristic 0, let $\alpha_1, \ldots, \alpha_n$ be nonzero elements of $\mathbf{K}$ and let Γ be a multiplicative subgroup of $(\mathbf{K}^*)^n$ of rank r. Then the equation*

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 1$$

has at most

$$e^{(6n)^{3n}(r+1)}$$

nondegenerate solutions $(x_1, \ldots, x_n) \in \Gamma$.

This theorem is the Main Theorem on S-unit equations over fields of characteristic 0. It is a generalization and refinement of earlier results due to Evertse and Győry [6], Evertse [4] and van der Poorten and Schlickewei [14] on the finiteness of the number of nondegenerate solutions of (3.1). For a general survey on these equations and their applications we refer to Evertse, Győry, Stewart and Tijdeman [7].

For the convenience of the reader, we state once more the consequence for the multiplicity of linear recurring sequences (see introduction, cf. [9]).

Theorem 3.2 (Evertse, Schlickewei and Schmidt). *Let $(u_m)_{m\in\mathbb{Z}}$ be a recurring sequence satisfying*

$$u_m = g_1\alpha_1^m + \cdots + g_n\alpha_n^m \quad \text{for } m \in \mathbb{Z},$$

where $\alpha_1,\dots,\alpha_n \in \mathbf{K}^*$ are distinct such that neither $\alpha_1,\dots,\alpha_n$, nor any of the quotients α_i/α_j $(1 \le i < j \le n)$ is a root of unity and where $g_1,\dots,g_n$ are nonzero elements of $\mathbf{K}$. Then for every $a \in \mathbf{K}$ we have

$$N(a) \le e^{(6n)^{3n}}.$$

Next we will consider equation (3.1) also over function fields. Let F be an algebraic function field in one variable with algebraically closed constant field $\mathbf{K}$ of characteristic 0. Thus F is a finite extension of $\mathbf{K}(t)$, where t is a transcendental element of F over $\mathbf{K}$. The field F can be endowed with a set M_F of additive valuations with value group $\mathbb{Z}$ for which

$$\mathbf{K} = \{0\} \cup \{z \in F \,|\, \nu(z) = 0 \text{ for each } \nu \text{ in } M_F\}$$

holds. Let S be a finite subset of M_F. An element z of F is called an *S-unit* if $\nu(z) = 0$ for all $\nu \in M_F\backslash S$. The S-units form a multiplicative group, which is denoted by U_S. The group U_S contains $\mathbf{K}^*$ as a subgroup and $U_S/\mathbf{K}^*$ is finitely generated. For function fields we have the following result:

Theorem 3.3 (Evertse and Győry). *Let $F,\mathbf{K},S$ be as above. Let g be the genus of $F/\mathbf{K}$, s the cardinality of S, and $n \ge 2$ an integer. Then for every $\alpha_1,\dots,\alpha_n \in F^*$, the set of solutions of*

$$\alpha_1x_1 + \cdots + \alpha_nx_n = 1 \text{ in } x_1,\dots,x_n \in U_S, \tag{3.3}$$

$$\text{with } \alpha_1x_1,\dots,\alpha_nx_n \text{ not all in } \mathbf{K} \tag{3.4}$$

is contained in the union of at most

$$\log(g+2)\cdot(e(n+1))^{(n+1)s+2}$$

$(n-1)$-dimensional linear subspaces of F^n.

For deriving this upper bound an effective upper bound of Brownawell and Masser [2] for the heights of solutions of (3.3) is used. For $n = 2$ the theorem gives the upper bound

$$\log(g+2)(3e)^{3s+2}$$

for the number of solutions of (3.3). We note that for the case $n = 2$ Evertse [5] established an upper bound, which is better and independent of g.

Theorem 3.4 (Evertse). *Let $F,\mathbf{K},S$ be as above. For each pair λ,μ in F^*, the equation*

$$\lambda x + \mu y = 1 \text{ in } x,y \in U_S$$

has at most $2 \cdot 7^{2s}$ *solutions with* $\lambda x/(\mu y) \notin \mathbf{K}$. *As above, s denotes the cardinality of* S.

We will use the results from above to prove the following proposition:

Proposition 3.5. *Let* $F, \mathbf{K}, S$ *be as above. Let* g *be the genus of* $F/\mathbf{K}$, s *the cardinality of* S, $n \geq 2$ *an integer, and* $\alpha_1, \ldots, \alpha_n \in F^*$. *Moreover, let* $\Gamma_i \subset U_S$, $i = 1, \ldots, n$ *and* $\mathcal{U} := \Gamma_1 \times \ldots \times \Gamma_n$. *We assume that any given pair* $(x_i, x_j) \in \Gamma_i \times \Gamma_j$ *with* $1 \leq i < j \leq n$ *gives rise to at most* k *solutions* $(x_1, \ldots, x_n)$ *of*

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 1 \text{ in } (x_1, \ldots, x_n) \in \mathcal{U}, \tag{3.5}$$

$$\text{where } \sum_{j \in J} \alpha_j x_j \neq 0 \text{ for each nonempty subset } J \text{ of } \{1, \ldots, n\}, \tag{3.6}$$

and that for arbitrary $\gamma_1, \gamma_2 \in F^*$ *there are at most* k *solutions* $(x_1, \ldots, x_n) \in \mathcal{U}$ *such that there exist indices* $i \neq j$ *with* $\gamma_1 x_i, \gamma_2 x_j \in \mathbf{K}^*$. *Then the number of solutions of* (3.5) *with* (3.6) *can by bounded by*

$$A(n, k) = k^n e^{n^2} (\log(g+2))^{n-2} (e(n+1))^{(n-1)(n+1)(s+1)}.$$

Finally, we need some results from the theory of algebraic function fields, which can be found, for example, in the monograph of Stichtenoth [19]. We will need the following estimate for the genus of a function field F/K (cf. [19], pp. 130 and 131).

Theorem 3.6 (Castelnuovo's Inequality)**.** *Let* F/K *be a function field with constant field* K. *Suppose there are given two subfields* F_1/K *and* F_2/K *of* F/K *satisfying*

(1) $F = F_1 F_2$ *is the compositum of* F_1 *and* F_2,
(2) $[F : F_i] = n_i$, *and* F_i/K *has genus* g_i $(i = 1, 2)$.

Then the genus g *of* F/K *is bounded by*

$$g \leq n_1 g_1 + n_2 g_2 + (n_1 - 1)(n_2 - 1).$$

We mention that Castelnuovo's Inequality is often sharp, and that, in general, it cannot be improved.

Moreover, we will use the Hurwitz Genus Formula (cf. [19], p. 88).

Theorem 3.7 (Hurwitz Genus Formula)**.** *Let* F/K *be an algebraic function field of genus* g *and let* F'/F *be a finite separable extension. Let* K' *denote the constant field of* F' *and* g' *the genus of* F'/K. *Then we have*

$$2g' - 2 = \frac{[F' : F]}{[K' : K]}(2g - 2) + \deg \operatorname{Diff}(F'/F).$$

The Hurwitz Genus Formula is a powerful tool that allows determination of the genus of F/K in terms of the different of $F/K(x)$ since any function field can be regarded as a finite extension of a rational function field.

Last we mention some basic facts about the valuation theory in function fields: Let $\mathbf{K}$ be an algebraically closed field of characteristic 0. Let K be a finite extension of $\mathbf{K}(x)$ where x is transcendental over $\mathbf{K}$. For $\xi \in \mathbf{K}$ define the valuation ν_ξ such that for $Q \in \mathbf{K}(x)$ we have $Q(x) = (x - \xi)^{\nu_\xi(Q)} A(x)/B(x)$ where A, B are polynomials with $A(\xi)B(\xi) \neq 0$. Furthermore, for $Q = A/B$ with $A, B \in \mathbf{K}[x]$ we put $\deg Q := \deg A - \deg B$; thus $\nu_\infty := -\deg$ is a discrete valuation on $\mathbf{K}(x)$. Each of the valuations ν_ξ, ν_∞ can be extended in at most $[K : \mathbf{K}(x)]$ ways to a discrete valuation on K and in this way one obtains all discrete valuations on K.

A valuation on K is called finite if it extends ν_ξ for some $\xi \in \mathbf{K}$ and infinite if it extends ν_∞. Let us mention that the valuations can be equivalently described by the concepts of places and valuation rings (cf. [19]).

4. Proof of Proposition 2.4[1]

The case when P is linear was treated in Remark 7 of our preceding paper [12]. This is exactly the second part of our assertion. Thus we assume in the sequel that $\deg P \geq 2$.

Assume that

$$A_0(x) = a\prod_{i=1}^{k}(x-a_i)^{n_i},$$

with pairwise different $a_1, \ldots, a_k$ and with positive $n_1, \ldots, n_k$. Since the roots of A_0 and of $A_0(P)$ are the same, we have

$$A_0(P(x)) = a\prod_{i=1}^{k}(P(x)-a_i)^{n_i} = a\,\mathrm{lc}(P)^{\deg A_0}\prod_{j=1}^{k}(x-a_j)^{m_j}$$

with nonzero $m_1, \ldots, m_k$ and where $\mathrm{lc}(P)$ denotes the leading coefficient of P. From this we get

$$P(x) - a_i = \mathrm{lc}(P)\prod_{j=1}^{k}(x-a_j)^{m_{ij}},$$

for all $i = 1, \ldots, k$, where the m_{ij} are nonnegative integers. If we assume that there exist indices $u \neq v$ with m_{uj}, m_{vj} both > 0, then we get that $a_v - a_u = \text{const}$ has a nontrivial divisor, namely $x - a_j$, contradicting the fact that $a_v - a_u$ is constant and different from zero.

Now we proceed as follows: Assume that we have

$$P(x) - a_1 = \mathrm{lc}(P)\prod_{j=1}^{k}(x-a_j)^{m_{1j}}.$$

There exists j_1 such that $m_{1j_1} > 0$ since $\deg P > 0$. From the discussion above this implies that $m_{ij_1} = 0$ for all $i = 2, \ldots, k$. Now look at

$$P(x) - a_2 = \mathrm{lc}(P)\prod_{\substack{j=1\\ j\neq j_1}}^{k}(x-a_j)^{m_{2j}}.$$

Now there exists $m_{2j_2} > 0$ and we have $m_{ij_2} = 0$ for $i = 1, 3, \ldots, k$, especially $m_{1j_2} = 0$. Continuing this we get, because there are k different roots, that there exists a permutation π of $\{1, \ldots, k\}$ such that

$$P(x) - a_i = \mathrm{lc}(P)(x - a_{\pi(i)})^{\deg P}, \quad \text{for } i = 1, \ldots, k. \tag{4.1}$$

If $k = 1$, then $A_0 = a(x-a_1)^{\deg A_0}$ and $\pi(1) = 1$; hence

$$P(x) = \mathrm{lc}(P)(x-a_1)^{\deg P} + a_1$$

and we obtain the first assertion of the proposition.

[1] We thank Ákos Pintér for helping us in a former proof of this proposition. The final part of the present argument was suggested by the anonymous referee, to whom we are also indebted.

It remains to prove that there are no more possibilities if the degree of P and the number of distinct zeros of A_0 are at least 2. Indeed, if $k \geq 2$, then (4.1) implies that

$$\frac{a_2 - a_1}{\mathrm{lc}(P)} = (x - a_{\pi(1)})^{\deg P} - (x - a_{\pi(2)})^{\deg P}.$$

We introduce a new variable $y = x - a_{\pi(2)}$ and put $e = a_{\pi(2)} - a_{\pi(1)}, f = (a_2 - a_1)/\mathrm{lc}(P)$, and $r = \deg P$. Then $(y+e)^r - y^r = f$; hence

$$\sum_{k=0}^{r-1} \binom{r}{k} e^{r-k} y^k = f,$$

which implies $r = 1, e = f$. Consequently, $\deg P$ must be one. $\square$

5. Proof of Proposition 3.5

We prove the assertion by induction on n. The case $n = 2$ follows easily from Theorem 3.3 for $n = 2$. Observe that by our assumptions there are at most k solutions with $\alpha_1 x_1$ and $\alpha_2 x_2$ both in $\mathbf{K}$. Therefore, we have at most

$$\log(g+2)(3e)^{3s+2} + k \leq A(2,k)$$

solutions $(x_1, x_2) \in \mathcal{U}$.

Now suppose $n > 2$ and that our claim has been shown for $n' < n$. Again by Theorem 3.3, either $\alpha_1 x_1, \dots, \alpha_n x_n$ all belong to $\mathbf{K}^*$, which by our assumption is possible for at most k solutions $(x_1, \dots, x_n)$, or $(x_1, \dots, x_n)$ lies in one of at most

$$\log(g+2) \cdot (e(n+1))^{(n+1)s+2}$$

proper linear subspaces of F^n.

Let $\mathcal{V}$ be one of these subspaces, defined by an equation

$$\gamma_1 x_1 + \cdots + \gamma_n x_n = 0$$

where $\gamma_i \in F$ for $i = 1, \dots, n$. Observe that at least two of the coefficients are different from zero. Without loss of generality we may assume that $\gamma_1 \neq 0$ and that $\gamma_1 = 1$, i.e.,

$$x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n = 0.$$

Subtracting this equation from our S-unit equation (3.5) under consideration gives

$$(\alpha_2 - \gamma_2 \alpha_1) x_2 + \cdots + (\alpha_n - \gamma_n \alpha_1) x_n = 1.$$

This is again an S-unit equation but now with $n-1$ variables. We write for the above equation

$$\sum_{i \in I} \delta_i x_i = 1$$

where I is a subset of $\{2, \dots, n\}$ of cardinality $|I| \geq 1$, because otherwise we would have

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = 0,$$

a contradiction to (3.5), and where $\delta_i \neq 0$ for $i \in I$. Let J be a nonempty subset of I and consider those solutions in $\mathcal{U} \cap \mathcal{V}$ for which

$$\sum_{i \in J} \delta_i x_i = 0, \tag{5.1}$$

but no proper nonempty subsum of (5.1) vanishes. Thus $1 \leq |J| \leq n-1$. We have to distinguish two cases, depending on whether $|J| = 1$ or $|J| \geq 2$.

Case 1. Let $J = \{u\}$ with $2 \leq u \leq n$. In this case the above equation reduces to

$$(\alpha_u - \gamma_u \alpha_1) x_u = 1,$$

with $\alpha_u \neq \gamma_u \alpha_1$. Therefore, we get

$$x_u = (\alpha_u - \gamma_u \alpha_1)^{-1},$$

and substituting this into (3.5) finally yields

$$\sum_{\substack{i=1 \\ i \neq u}}^{n} \frac{\alpha_i}{(1 - (\alpha_u - \gamma_u \alpha_1)^{-1} \alpha_u)} x_i = 1. \tag{5.2}$$

Observe that the denominator is different from 0, because otherwise we would have

$$\sum_{\substack{i=1 \\ i \neq u}}^{n} \alpha_i x_i = 0,$$

leading to a contradiction to assumption (3.6). Equation (5.2) is an S-unit equation with $n-1$ variables. By induction, we can conclude that this equation has at most $A(n-1,k)$ solutions such that no nontrivial subsum vanishes, and since each of these solutions gives rise to at most k solutions of (3.5), we conclude that we get at most $kA(n-1,k)$ solutions $(x_1, \ldots, x_n)$ in this case. Observe that any vanishing subsum of equation (5.2) would immediately lead to a vanishing subsum of our original equation, and thus we do not have to take them into account.

Case 2. Now we can assume that $|J| \geq 2$ and in this situation we can at once use the induction hypothesis. Thus we conclude that (5.1) has at most $A(n-1,k)$ solutions, where no subsum vanishes. Observe that the vanishing subsums of

$$\sum_{i \in I} \delta_i x_i = 1$$

are taken into account by the different choices of J. Since we know by our assumptions that each of these solutions gives rise to at most k solutions of our original problem, we get at most $kA(n-1,k)$ solutions $(x_1, \ldots, x_n)$ also in this case.

By consideration of the possible subsets J of I, we see that each subspace $\mathcal{V}$ contains at most $2^n kA(n-1,k)$ solutions. We still have to multiply this by the number of subspaces. In this way we obtain a bound

$$2^n kA(n-1,k) \log(g+2)(e(n+1))^{(n+1)s+2} + k.$$

This is

$$\begin{aligned}
2^{n+1} k k^{n-1} e^{(n-1)^2} & (\log(g+2))^{n-3} (en)^{(n-2)n(s+1)} \cdot \log(g+2)(e(n+1))^{(n+1)s+2} \\
&\leq k^n e^{n^2-n+2} (\log(g+2))^{n-2} \cdot (e(n+1))^{(n-2)(n+1)(s+1)} (e(n+1))^{(n+1)(s+1)} \\
&\leq A(n,k),
\end{aligned}$$

and, therefore, Proposition 3.5 follows. $\square$

6. Preliminaries and properties of the field of definition

Let $(P_n(x))_{n=0}^{\infty}$ be defined by (1.1). Moreover, let $\alpha_1(x), \dots, \alpha_d(x)$ as well as $\alpha_1(P(x)), \dots, \alpha_d(P(x))$ denote the roots of the characteristic polynomial of $(G_n(x))_{n=0}^{\infty}$ and $(G_n(P(x))_{n=0}^{\infty}$ respectively. We will always assume that $D(x) \neq 0$ (which follows from condition (ii) in all theorems); thus $\alpha_1(x), \dots, \alpha_d(x)$ are pairwise distinct. In the above argument we may replace x by any other element that is transcendental over $\mathbf{K}$. So, in particular, $\alpha_1(P(x)), \dots, \alpha_d(P(x))$ are pairwise distinct.

Let us define

$$F = \mathbf{K}(x, \alpha_1(x), \dots, \alpha_d(x), \alpha_1(P(x)), \dots, \alpha_d(P(x))).$$

Then F is a finite extension of $\mathbf{K}(x)$, i.e., we have an algebraic function field in one variable over the constant field $\mathbf{K}$. Denote as usual the multiplicative group of F by F^*. Define Γ to be the subgroup of F^* generated by

$$\alpha_1(x), \dots, \alpha_d(x), \alpha_1(P(x)), \dots, \alpha_d(P(x));$$

these are the characteristic roots of $(G_n(x))_{n=0}^{\infty}$ and $(G_n(P(x)))_{n=0}^{\infty}$, respectively.

It is obvious that Γ can be seen as a finitely generated subgroup of $\mathbb{C}^*$, because we can embed F^* into $\mathbb{C}^*$ by choosing a maximal set of algebraically independent elements from x and the coefficients of $A_0, \dots, A_{d-1}, G_0, \dots, G_{d-1}$, and sending the elements of this set to algebraically independent elements of $\mathbb{C}$. Moreover, it is clear that the rank r of Γ is at most $2d$.

F will be the field of definition for our problem, because we will reduce the equations under consideration to linear equations over F, where we look for solutions in Γ. First, we will deduce some more information about these sets and we will do this in the following lemmas.

First, we calculate the genus of the function field $F/\mathbf{K}$.

Lemma 6.1. *We denote by g the genus of the function field $F/\mathbf{K}$. Then we have*

$$g \leq d^{2d^2} \deg D(\deg P + 1) - 2.$$

Proof. First observe that we have

$$F = \mathbf{K}(x)(\alpha_1(x), \dots, \alpha_d(x)) \cdot \mathbf{K}(x)(\alpha_1(P(x)), \dots, \alpha_d(P(x))).$$

Let us denote

$$F_1 = \mathbf{K}(x, \alpha_1(x), \dots, \alpha_d(x)), \qquad F_2 = \mathbf{K}(x, \alpha_1(P(x)), \dots, \alpha_d(P(x))).$$

Furthermore, we denote by g_i the genus of $F_i/\mathbf{K}$ $(i = 1, 2)$. We have

$$n_1 = [F : F_1] \leq d! \quad \text{and} \quad n_2 = [F : F_2] \leq d!.$$

Next, we calculate bounds for g_1, g_2. Observe that

$$F_1 = \mathbf{K}(x, \alpha_1(x)) \cdot \mathbf{K}(x, \alpha_2(x)) \cdots \mathbf{K}(x, \alpha_d(x)).$$

We apply the Hurwitz Genus Formula (Theorem 3.7) to $\mathbf{K}(x)/\mathbf{K}$ and $\tilde{F}/\mathbf{K}(x)$, where $\tilde{F} = \mathbf{K}(x, \alpha_i(x))$ for some $i = 1, \dots, d$. Observe that the constant field of F and, therefore, of all intermediate fields is $\mathbf{K}$ and that the genus of the rational function field $\mathbf{K}(x)$ is zero (cf. [19], p. 22). Denote by $\tilde{g}$ the genus of $\tilde{F}$. Therefore, we get

$$2\tilde{g} - 2 = -2[\tilde{F} : \mathbf{K}(x)] + \deg \operatorname{Diff}(\tilde{F}/\mathbf{K}(x)).$$

We calculate the different:

$$\mathrm{Diff}(\tilde{F}/\mathbf{K}(x)) = \sum_{P\in\mathbb{P}_{\mathbf{K}(x)}} \sum_{P'|P} d(P'|P)P',$$

where $\mathbb{P}_{\mathbf{K}(x)}$ denotes the set of places of $\mathbf{K}(x)$ and $P'|P$ means that $P' \in \mathbb{P}_{\tilde{F}}$ (the places of $\tilde{F}$) lies over P. The second sum is extended over all extensions of P. Because of $\operatorname{char}\mathbf{K} = 0$, we conclude by Dedekind's Different Theorem (cf. [19], p. 89) that

$$d(P'|P) = e(P'|P) - 1 \le e(P'|P),$$

for all places P of $\mathbf{K}(x)$ and for all places P' in $\tilde{F}$ lying over P and where $e(P'|P)$ denotes the ramification index. Moreover, for almost all P and P' we have $d(P'|P) = 0$, i.e., $e(P'|P) = 1$. So, let us denote by S the set of places $P \in \mathbb{P}_{\mathbf{K}(x)}$ that ramify in $\tilde{F}$.

We calculate

$$\deg \mathrm{Diff}(\tilde{F}/\mathbf{K}(x)) = \sum_{P\in S}\sum_{P'|P} d(P'|P)\deg P' \le \sum_{P\in S}\sum_{P'|P} e(P'|P)\deg P'.$$

We use

$$\deg P' = [\tilde{F}_{P'} : \mathbf{K}] = [\tilde{F}_{P'} : \mathbf{K}(x)_P]\cdot[\mathbf{K}(x)_P : \mathbf{K}] = f(P'|P)\cdot \deg P,$$

where $\tilde{F}_{P'}, \mathbf{K}(x)_P$ are the residue class fields of P', P respectively and $f(P'|P)$ is the relative degree of P' over P. Thus

$$\deg \mathrm{Diff}(\tilde{F}/\mathbf{K}(x)) \le \sum_{P\in S}\deg P \sum_{P'|P} e(P'|P)f(P'|P) = \sum_{P\in S}\deg P\cdot[\tilde{F} : \mathbf{K}(x)],$$

where we have used Theorem III.1.11 in [19] to get the last equation. Finally, we use that, since $\mathbf{K}$ is algebraically closed, $\deg P = 1$ for all $P \in S$ (e.g., cf. Proposition I.2.1 in [19]) and that $[\tilde{F} : \mathbf{K}(x)] \le d$. This implies that $\deg \mathrm{Diff}(\tilde{F}/\mathbf{K}(x)) \le d\,|S|$ and finally,

$$\tilde{g} \le \frac{|S|-2}{2}d + 1 \le d(|S|-1),$$

where $|S|$ denotes as usual the cardinality of S and where we have assumed that $d \ge 2, |S| \ge 2$, which is no loss of generality.

We want to bound $|S|$. Observe that $\alpha_1(x), \ldots, \alpha_d(x)$ are integral over $\mathbf{K}[x]$ since they are the roots of a monic polynomial with coefficients in $\mathbf{K}[x]$ (namely the characteristic polynomial). Therefore, there exists an—over $\mathbf{K}(x)$—monic irreducible polynomial $H(T)$ with coefficients in $\mathbf{K}[x]$ which generates $\tilde{F}/\mathbf{K}(x)$. For $\xi \in \mathbf{K}$ let P_ξ denote the place in $\mathbf{K}(x)$ corresponding to $x - \xi$. We have $e(P|P_\xi) = 1$ for almost all $\xi \in \mathbf{K}$. Indeed, by a theorem of Kummer (cf. [19], p. 80) only at the poles of the coefficients of the polynomial $H(T)$ and at roots of the discriminant of $H(T)$ ramifications may occur (and the coefficients are polynomials and thus do not have poles at finite places). Since $H(T)$ divides $Q(T)$, the discriminant of $H(T)$ must divide $D(x)$. Hence,

$$|S| \le \deg D + 1$$

and, therefore,

$$\tilde{g} \le d \deg D.$$

Now we use Castelnuovo's Inequality (Theorem 3.6) several times and conclude that

$$\begin{aligned} g_1 &\leq (\ldots(((2d^2 \deg D + d^2)d^2 + d^2 \deg D + d^3)d^3 + d^2 \deg D + d^4)d^4 \\ &\qquad\qquad\qquad\qquad\qquad + \ldots)d^{d-1} + d^2 \deg D + d^d \\ &\leq d^{d^2}(\deg D + 1). \end{aligned}$$

In precisely the same way, we can conclude that

$$g_2 \leq d^{d^2}(\deg D \deg P + 1).$$

Now using Castelnuovo's Inequality (Theorem 3.6) once again we get

$$\begin{aligned} g &\leq d!d^{d^2}(\deg D + 1) + d!d^{d^2}(\deg D \deg P + 1) + d!^2 \\ &\leq d!d^{d^2}(\deg D(\deg P + 1) + 3) \leq d^{2d^2} \deg D(\deg P + 1) - 2, \end{aligned}$$

and, therefore, our proof is finished. □

Next, we prove the following lemma:

Lemma 6.2. *We assume* $\deg A_0 \geq 1$ *and* $\deg P \geq 1$. *Then there exists a finite subset* $S \subset M_F$ *of valuations of the function field* F *such that* Γ *is contained in the group of* S*-units* U_S *and such that*

$$|S| \leq d!^2(\deg A_0(\deg P + 1) + 1) \leq 6d!^2 \deg A_0 \deg P - 1.$$

Proof. Let S_∞ be the set of infinite valuations of F, and S_0 the set of finite valuations of F. Note that for every $\nu \in S_0$ we have $\nu(\alpha_1) \geq 0, \ldots, \nu(\alpha_d) \geq 0, \nu(\alpha_1(P)) \geq 0, \ldots, \nu(\alpha_d(P)) \geq 0$ since these functions are integral over $\mathbf{K}[x]$. Take

$$S = S_\infty \cup \bigcup_{i=1}^{2d} S_i,$$

where

$$S_i = \{\nu \in S_0 \mid \nu(\alpha_i) > 0\}, \qquad S_{d+i} = \{\nu \in S_0 \mid \nu(\alpha_i(P)) > 0\}$$

for $i = 1, \ldots, d$. Then clearly Γ is a subgroup of U_S. Since $[F : \mathbf{K}(x)] \leq d!^2$, we have $|S_\infty| \leq d!^2$. Furthermore,

$$\alpha_1(x) \cdots \alpha_d(x) \cdot \alpha_1(P(x)) \cdots \alpha_d(P(x)) = A_0(x) \cdot A_0(P(x)) =: Q(x).$$

Therefore,

$$\bigcup_{i=1}^{2d} S_i =: \tilde{S} := \{\nu \in S_0 \mid \nu(Q) > 0\}.$$

Each of the valuations in $\tilde{S}$ is an extension to F of some valuation ν_ξ on $\mathbf{K}(x)$ corresponding to a zero ξ of $Q(x)$. The polynomial $Q(x)$ has at most $\deg Q = \deg A_0(\deg P + 1)$ zeros, and for each of these zeros ξ, the valuation ν_ξ can be extended in at most $d!^2$ ways to a valuation on F. Therefore,

$$|\tilde{S}| \leq d!^2 \deg A_0(\deg P + 1).$$

This implies

$$\begin{aligned} |S| &\leq d!^2(\deg A_0(\deg P + 1) + 1) \\ &\leq d!^2(2 \deg A_0 \deg P + 1) \leq 6d!^2 \deg A_0 \deg P - 1, \end{aligned}$$

since $\deg A_0 \geq 1$ and $\deg P \geq 1$, which was our assertion. □

Finally, we need the following properties:

Lemma 6.3. *Assume that none of the roots and the quotient of distinct roots of the characteristic polynomial of $(G_n(x))_{n=0}^{\infty}$ is an element of $\mathbf{K}^*$. Let γ_1, γ_2 be nonzero elements of F. Then there is at most one pair of integers n, m such that*

$$\gamma_1 \frac{\alpha_i(x)^n}{\alpha_k(P(x))^m} \in \mathbf{K}^* \quad \text{and} \quad \gamma_2 \frac{\alpha_j(x)^n}{\alpha_k(P(x))^m} \in \mathbf{K}^* \tag{6.1}$$

or

$$\gamma_1 \frac{\alpha_i(x)^n}{\alpha_k(P(x))^m} \in \mathbf{K}^* \quad \text{and} \quad \gamma_2 \frac{\alpha_j(P(x))^m}{\alpha_k(P(x))^m} \in \mathbf{K}^*, \tag{6.2}$$

respectively, where $1 \le i, j, k \le d$ are different integers.

Proof. First we prove equation (6.1). Suppose there are two such pairs (n_1, m_1), (n_2, m_2). Let $n = n_1 - n_2, m = m_1 - m_2$. Then, by dividing the first equations ((6.1) with n_1, m_1) by the second equations ((6.1) with n_2, m_2) we get

$$\frac{\alpha_i(x)^n}{\alpha_k(P(x))^m} \in \mathbf{K}^* \quad \text{and} \quad \frac{\alpha_j(x)^n}{\alpha_k(P(x))^m} \in \mathbf{K}^*; \tag{6.3}$$

hence $\alpha_i(x)^n / \alpha_j(x)^n \in \mathbf{K}^*$. But this can only hold if $\alpha_i(x)/\alpha_j(x) \in \mathbf{K}^*$, which contradicts our assumption, or if $n = 0$, whence $n_1 = n_2$ and so by (6.3) we get also $m_1 = m_2$.

In the same way, if we assume that (6.2) holds for two such pairs, we conclude that

$$\frac{\alpha_i(x)^n}{\alpha_k(P(x))^m} \in \mathbf{K}^* \quad \text{and} \quad \frac{\alpha_j(P(x))^m}{\alpha_k(P(x))^m} \in \mathbf{K}^*. \tag{6.4}$$

But now we can conclude that $m = 0$ or $m_1 = m_2$ by using the second part of (6.4) and then by the first part of (6.4), we get also $n_1 = n_2$. Observe that we have used our assumption twice to get this. □

In the next section, we will reduce the solvability of our equation (2.3) to the solvability of a system of critical exponential equations in n, m.

7. Reduction to a system of equations

First observe that by (2.2) we have $g_i(x), g_i(P(x)) \in F$ for $i = 1, \ldots, d$. The same equation and the definition of R implies that

$$\prod_{j=1}^{d} g_j(x)\alpha_j(x) \prod_{\substack{i=1 \\ i \ne j}}^{d} (\alpha_i(x) - \alpha_j(x)) = R.$$

By condition (iv) of Theorems 2.1 and 2.5 or condition (iii) of Theorems 2.3 and 2.7, we have $R \ne 0$. Hence $g_j(x) \ne 0$ for $j = 1, \ldots, d$. This ensures that $g_j(P(x)) \ne 0$ for $j = 1, \ldots, d$.

Assume that $n, m \ge 0, n \ne m$ are integers satisfying $G_n(x) = c\, G_m(P(x))$ for some $c \in \mathbf{K}^*$. It follows that

$$\sum_{i=1}^{d} g_i(x)\alpha_i(x)^n = c \sum_{i=1}^{d} g_i(P(x))\alpha_i(P(x))^m. \tag{7.1}$$

We have already seen that $g_d(P(x)) \ne 0$. We have $A_0 \ne 0$ by (ii) and (iii) in Theorems 2.1 and 2.5 and by (iii) in Theorems 2.3 and 2.7, respectively; hence

$\alpha_d(P(x)) \neq 0$ holds also. Dividing by $g_d(P(x))\alpha_i(P(x))^m$ and sorting the summands we obtain the weighted equation

$$\sum_{i=1}^{d} \frac{g_i(x)}{g_d(P(x))} x_i - \sum_{i=1}^{d-1} \frac{g_i(P(x))}{g_d(P(x))} x_{d+i} = 1 \tag{7.2}$$

in the unknowns

$$x_j = c^{-1} \frac{\alpha_j(x)^n}{\alpha_d(P(x))^m} \text{ for } j = 1, \dots, d,$$
$$x_{d+j} = \frac{\alpha_j(P(x))^m}{\alpha_d(P(x))^m} \text{ for } j = 1, \dots, d-1.$$

Observe that $x_1, \dots, x_{2d-1}$ are elements of the set U_S, which exists by Lemma 6.2. This is because of the fact that Γ is contained in U_S and $c \in \mathbf{K}^*$. Lemma 6.3 implies that any given pair of elements (x_i, x_j) or (x_i, x_{d+j}) for $1 \leq i < j \leq d$ gives rise to at most one pair (n, m); especially any tuple $(x_1, \dots, x_{2d-1})$ induces at most one solution (n, m) of the equation under consideration. Because of the fact that $\alpha_1(x), \dots, \alpha_d(x)$ are not in $\mathbf{K}^*$ (and therefore also $\alpha_1(P(x)), \dots, \alpha_d(P(x))$ are not in $\mathbf{K}^*$) it follows that a given pair (x_{d+i}, x_{d+j}) $(1 \leq i < j \leq d)$ induces at most one m. We will show that this in turn induces (via our equation $G_n(x) = c\, G_m(P(x))$ at most finitely many pairs (n, m).

We set

$$\beta_i = \frac{g_i(x)}{g_d(P(x))} \quad \text{and} \quad \beta_{d+i} = \frac{g_i(P(x))}{g_d(P(x))} \quad \text{for} \quad i = 1, \dots, d.$$

Now let us assume that we can bound the number of solutions of the equation

$$\beta_1 x_1 + \cdots + \beta_d x_d + \beta_{d+1} x_{d+1} + \cdots + \beta_{2d-1} x_{2d-1} = 1,$$

where no nontrivial subsum vanishes, by a constant $W(2d-1)$. More generally we assume that the number of solutions of

$$\gamma_1 y_1 + \cdots + \gamma_n y_n = 1, \tag{7.3}$$

with $\gamma_1, \dots, \gamma_n \in F^*$ and $y_i = x_j$ or $= x_{d+j}$ for some j, where no nontrivial subsum vanishes and which leads to a solution of our equation, can be bounded by $W(n)$. This is, of course, true in the special case that c is always equal to 1 by the theorem of Evertse, Schlickewei and Schmidt (Theorem 3.1). In the more general case we will deduce this later (see Section 9) using Proposition 3.5.

Let J be a nonempty subset of $\{1, \dots, 2d-1\}$ with $1 \leq |J| \leq 2d-2$ and consider those solutions $(x_1, \dots, x_{2d-1}) \in U_S^{2d-1}$ of the above equation (7.2) for which

$$\sum_{i \in J} \beta_i x_i = 1, \tag{7.4}$$

but no proper nonempty subsum of (7.4) vanishes. We have to distinguish three cases:

Case 1. First we assume that $|J| = 2d-2$. In this case we must have $\beta_j x_j = 0$ for the single j not belonging to J. But this cannot hold since $\beta_j \neq 0$ for $j = 1, \dots, d$ and $0 \notin \Gamma \subset U_S$.

Case 2. The second case is that $J \subseteq \{d+1, \dots, 2d-1\}$. This case is special because the components of (7.4) now depend only on m. By Theorem 3.2 we obtain that (7.4) has at most

$$e^{(6(d-1))^{3(d-1)}}$$

solutions. This implies that we have at most that many possibilities for m. For fixed m the right-hand side of

$$\sum_{i=1}^{d} g_i(x) c^{-1} \alpha_i(x)^n = \sum_{i=1}^{d} g_i(P(x)) \alpha_i(P(x))^m$$

is a fixed element, namely $G_m(P(x))$, of $\mathbf{K}[x]$. If $G_m(P(x)) = 0$, then we obtain $G_n(x) = 0$, which can hold by Theorem 3.2 for at most

$$e^{(6d)^{3d}}$$

many n also. Otherwise, dividing by $G_m(P(x))$ we get

$$\sum_{i=1}^{d} \frac{g_i(x)}{G_m(P(x))} y_i = 1,$$

where

$$y_i = c^{-1} \alpha_i(x)^n \text{ for } i = 1, \dots, d.$$

This is again a weighted S-unit equation, whose number of solutions we can bound by $W^*(d)$, which we will again show later. Taking account of the possible subsets J, we see that we have at most

$$2^d \max\left\{W^*(d), e^{(6d)^{3d}}\right\} e^{(6(d-1))^{3(d-1)}}$$

pairs of solutions (n, m) in this case.

Case 3. The remaining case is $J \cap \{1, \dots, d\} \neq \emptyset$. We consider two subcases: The first subcase is $1 < |J| \leq 2d-3$. In this case we can bound the number of solutions (n, m) by $W(2d-3)$ since (7.4) is a weighted S-unit equation with $2d-3$ variables. The number of cases can be bounded by 4^d. Thus we have

$$4^d W(2d-3)$$

possible solutions. The last subcase is $|J| = 1$. Since we have $J \cap \{1, \dots, d\} \neq \emptyset$, we conclude that $\beta_u x_u = 1$ for some $1 \leq u \leq d$, i.e., we have

$$g_u(x) \alpha_u(x)^n = c\, g_d(P(x)) \alpha_d(P(x))^m.$$

If this is true, then the equation

$$\sum_{\substack{i=1 \\ i \neq u}}^{d} g_i(x) \alpha_i(x)^n = c \sum_{\substack{j=1 \\ j \neq d}}^{d} g_j(P(x)) \alpha_j(P(x))^m$$

must simultaneously hold. But this is essentially the same equation as (7.1) with one summand less at both sides of the equation. Thus, we can continue this process and ultimately obtain that the equation (7.1) has at most

$$d \cdot \left(W(2d-1) + 2^d \max\left\{W^*(d), e^{(6d)^{3d}}\right\} e^{(6(d-1))^{3(d-1)}} + 4^d W(2d-3)\right)$$

solutions $(n,m) \in \mathbb{Z}^2, n,m \geq 0, n \neq m$ or it is a solution of a system of equations of the form

$$g_u(x)\alpha_u(P(x))^n = c\, g_{\pi(u)}(P(x))\alpha_{\pi(u)}(P(x))^m, \quad u = 1,\dots,d,$$

where π is a permutation of the set $\{1,\dots,d\}$.

To handle this exceptional system of equations, we will need most of the assumptions in our theorems. We will handle these cases in the next section.

8. Handling the exceptional cases

We start with the system of equations

$$g_u(x)\alpha_u(P(x))^n = c\, g_{\pi(u)}(P(x))\alpha_{\pi(u)}(P(x))^m, \quad u = 1,\dots,d, \tag{8.1}$$

where π is a permutation of the set $\{1,\dots,d\}$. We will show that this system has only finitely many solutions (n,m).

First we assume the conditions of Theorems 2.1 and 2.5. Indeed, in this case we have $\deg D(P) = \deg D \deg P > \deg D \geq 1$, since $\deg P > 1$ by assumption (ii). On the other hand, we have

$$D(P(x)) = \prod_{j=1}^{d} Q(P)'(\alpha_j(P(x))) \quad \text{and} \quad D(x) = \prod_{j=1}^{d} Q'(\alpha_j(x)),$$

where $'$ denotes differentiation with respect to the variable T. Hence there exists a pair $(u,v) = (u,\pi(u))$ and a finite valuation ν of F such that

$$\nu(Q(P)'(\alpha_v(P))) > \nu(Q'(\alpha_u)) \geq 0.$$

Before continuing we state the following useful lemma.

Lemma 8.1. *Let $A, B, P \in \mathbf{K}[x]$. Then*

$$\gcd(A,B) = 1 \textit{ if and only if } \gcd(A(P), B(P)) = 1.$$

This lemma is a special case of a lemma in the monograph of Schinzel [16], p. 16. It was originally proved in [15].

Now assumption (iii) of Theorem 2.1 together with Lemma 8.1 implies that

$$\nu(\alpha_v(P)) = 0,$$

since $\nu(D(P)) > 0$, while assumption (iv) implies that

$$\nu\left(\sum_{i=0}^{d-1} L_i(\vec{A},\vec{G})\alpha_v(P)^i\right) = 0.$$

Hence (2.2) implies (with v instead of 1) that

$$\nu(g_v(P)) = -\nu\left(\prod_{\substack{i=1\\ i\neq v}}^{d}(\alpha_i(P) - \alpha_v(P))\right) = -\nu(Q(P)'(\alpha_v(P))).$$

Therefore, (8.1) implies that

$$\nu(g_u) + n\nu(\alpha_u) = -\nu(Q(P)'(\alpha_v(P))), \tag{8.2}$$

where we have used $\nu(c) = 0$ since $c \in \mathbf{K}^*$. Observe that $\alpha_1(x),\dots,\alpha_d(x)$ are integral over $\mathbf{K}[x]$, since they are zeros of the monic characteristic equation $Q(T) = 0$ with coefficients in $\mathbf{K}[x]$. The integral closure of $\mathbf{K}[x]$ in F consists of those elements f such that $\nu(f) \geq 0$ for every finite valuation ν of F. So, in particular, $\nu(\alpha_u) \geq 0$.

Using (2.2) once again (with u instead of 1), we get

$$\nu(g_u) + \nu(\alpha_u) + \nu(Q'(\alpha_u)) = \nu\left(\sum_{i=0}^{d-1} L_u(\vec{A},\vec{G})\alpha_u^i\right) \geq 0,$$

by the remark from above and the fact that $L_u(\vec{A},\vec{G})(x) \in \mathbb{Q}[\vec{A},\vec{G}]$ and the components of $\vec{A}$ and $\vec{G}$ are (as polynomials) integral elements. Therefore, we conclude that

$$\nu(g_u) + \nu(\alpha_u) \geq -\nu(Q'(\alpha_u)),$$

which yields, together with (8.2),

$$\begin{aligned} n\nu(\alpha_u) &= -\nu(Q(P)'(\alpha_v(P))) - \nu(g_u) \\ &\leq -\nu(Q(P)'(\alpha_v(P))) + \nu(Q'(\alpha_u)) + \nu(\alpha_u) < \nu(\alpha_u). \end{aligned}$$

Since $\nu(\alpha_u) \geq 0$, we conclude that $n = 0$. Thus, equation (8.1) induces only solutions of the kind $(0, m)$ with $m > 0$.

Now we have to distinguish between the assumptions of Theorems 2.1 and 2.5. First let us assume (i) of Theorem 2.1. We investigate for $n = 0$,

$$g_u(x) = c\, g_v(P(x))\alpha_v(P(x))^m.$$

Assume that there are two solutions m_1 and m_2. Then we have

$$c_1\alpha_v(P(x))^{m_1} = c_2\alpha_v(P(x))^{m_2}$$

($c \in \mathbf{K}^*$ depends on m) or

$$\alpha_v(P(x))^{m_1-m_2} \in \mathbf{K}^*,$$

which is a contradiction unless $m_1 = m_2$. Therefore, we get for each system (8.1) at most one solution (n, m).

Now assume that we have assumption (i) of Theorem 2.5. By the arguments above one has

$$\alpha_v(P(x))^{m_1-m_2} = 1,$$

which implies $m_1 = m_2$. So there is only one possible value for m and, therefore, at most one solution (n, m).

We consider now the assumptions of Theorems 2.3 and 2.7. From condition (iv) (together with (ii) and (iii)) we get that there is a valuation ν of F with

$$\nu(A_0) > 0 \text{ and } \nu(A_0(P)) = 0 \quad \text{or} \quad \nu(A_0) = 0 \text{ and } \nu(A_0(P)) > 0.$$

From this we can conclude (observe that $\alpha_1(x), \ldots, \alpha_d(x)$ are integral over $\mathbf{K}[x]$) that there exists an index u ($1 \leq u \leq d$) with either

$$\nu(\alpha_u) > 0 \quad \text{and} \quad \nu(\alpha_i(P)) = 0 \quad \text{for } i = 1, \ldots, d,$$

or

$$\nu(\alpha_u(P)) > 0 \quad \text{and} \quad \nu(\alpha_i) = 0 \quad \text{for } i = 1, \ldots, d.$$

In the first case, we look at the equation

$$g_u(x)\alpha_u(x)^n = c\, g_{\pi(u)}(P(x))\alpha_{\pi(u)}(P(x))^m$$

of the system (8.1). We get

$$\nu(g_u) + n\nu(\alpha_u) = \nu(g_{\pi(u)}).$$

But this can hold for at most one n, namely

$$n_0 = \frac{\nu(g_{\pi(u)}) - \nu(g_u)}{\nu(\alpha_u)}.$$

In the second case, we look at

$$g_{\pi^{-1}(u)}(x)\alpha_{\pi^{-1}(u)}(x)^n = c\, g_u(P(x))\alpha_u(P(x))^m.$$

Similarly as above, we get

$$\nu(g_{\pi^{-1}(u)}) = \nu(g_u(P)) + m\nu(\alpha_u(P))$$

and this can only hold for at most one m, namely

$$m_0 = \frac{\nu(g_{\pi^{-1}(u)}) - \nu(g_u(P))}{\nu(\alpha_u(P))}.$$

Let us first assume the conditions of Theorem 2.3. The case that there is at most one $n = n_0$ implies as above that we have

$$c\,\alpha_{\pi(u)}(P(x))^m = \frac{g_u(x)\alpha_u(x)^{n_0}}{g_{\pi(u)}(P(x))}.$$

From this we can again conclude that there is at most one m also. The case that there is at most one $m = m_0$ runs along the same line and gives by

$$c^{-1}\alpha_{\pi^{-1}(u)}(x)^n = \frac{g_u(P(x))\alpha_u(P(x))^{m_0}}{g_{\pi^{-1}(u)}(x)}$$

and condition (i) that there is at most one n. So in this case each system (8.1) gives at most one solution (n, m).

Finally, we assume the hypotheses of Theorem 2.7. Again, as above, we get via the equations

$$\alpha_{\pi(u)}(P(x))^{m_1-m_2} = 1 \quad \text{or} \quad \alpha_{\pi^{-1}(u)}(x)^{n_1-n_2} = 1$$

that there is at most one possible solution (n, m) in both cases.

To sum up we can bound the number of solutions $(n, m) \in \mathbb{Z}^2, n, m \geq 0, n \neq m$ that come from systems of equations (8.1) by

$$d!.$$

Using this bound and the other bound calculated in the previous section we will be able to complete our proof and we will do this for Theorems 2.1 and 2.3 and Theorems 2.5 and 2.7 in separate sections.

9. Calculation of the bounds for Theorems 2.1 and 2.3

It is left to show that the equation (cf. (7.3))

$$\gamma_1 y_1 + \cdots + \gamma_n y_n = 1$$

with $\gamma_1, \ldots, \gamma_n \in F^*$ and

$$y_i = c^{-1}\frac{\alpha_i(x)^n}{\alpha_d(P(x))^m} \quad \text{or} \quad y_i = \frac{\alpha_i(P(x))^m}{\alpha_d(P(x))^m}$$

for some i, where no nontrivial subsum vanishes and which leads to a solution of $G_n(x) = c\,G_m(P(x))$ has at most $W(n)$ solutions.

We first show this for the equation

$$\sum_{i=1}^{d} \frac{g_i(x)}{g_d(P(x))} x_i - \sum_{i=1}^{d-1} \frac{g_i(P(x))}{g_d(P(x))} x_{d+i} = 1$$

where $x_1, \ldots, x_{2d-1}$ are elements of the set $\mathbf{K}\Gamma \subset U_S$. Take $\mathcal{U}$ to be the subset of $(\mathbf{K}\Gamma)^{2d-1}$ consisting of elements $(x_1, \ldots, x_{2d-1})$ of the same form as above which lead to solutions of $G_n(x) = c\, G_m(P(x))$ where S is the set of absolute values of F from Lemma 6.2. We have to show that we have at most $W(2d-1)$ solutions where no nontrivial subsum vanishes and we do this by applying Proposition 3.5. Therefore, we first have to show that each pair (x_i, x_j) gives rise to at most k solutions of the above equation. Lemma 6.3 implies that any given pair of elements (x_i, x_j) or (x_i, x_{d+j}) for $1 \le i < j \le d$ gives rise to at most one pair (n, m). Because of the fact that $\alpha_1(x), \ldots, \alpha_d(x)$ are not in $\mathbf{K}^*$ (and, therefore, also $\alpha_1(P(x)), \ldots, \alpha_d(P(x))$ are not in $\mathbf{K}^*$) it follows that a given pair (x_{d+i}, x_{d+j}) $(1 \le i < j \le d)$ induces at most one m. But this implies that we have to calculate the number of solutions n of $G_n(x) = c\, G_m(P(x))$. If $G_m(P(x)) = 0$, this equation reduces to $G_n(x) = 0$, which can happen by Theorem 3.2 for at most

$$e^{(6d)^{3d}}$$

many n. Now, if $G_m(P(x))$ is different from 0, we have to consider the equation

$$\frac{g_1(x)}{G_m(P(x))} z_1 + \cdots + \frac{g_d(x)}{G_m(P(x))} z_d = 1, \tag{9.1}$$

where

$$z_1 = c^{-1}\alpha_1(x)^n,\ \ldots,\ z_d = c^{-1}\alpha_d(x)^n.$$

But we can apply Proposition 3.5 (where we put $\mathcal{U} = \{(c^{-1}\alpha_1(x)^n, \ldots, c^{-1}\alpha_d(x)^n) \in (\mathbf{K}\Gamma)^n\}$) to this equation once more: each pair (z_i, z_j) gives rise to at most one n, because otherwise we have $\alpha_i(x)/\alpha_j(x) \in \mathbf{K}^*$, which contradicts assumption (i) in Theorems 2.1 and 2.3. Moreover, assume that we have

$$\gamma c^{-1}\alpha_i(x)^{n_1} \in \mathbf{K}^*,\ \gamma c^{-1}\alpha_i(x)^{n_2} \in \mathbf{K}^*,$$

where $\gamma \in F^*$. Then we get a contradiction unless $n_1 = n_2$. Thus we can bound the number of solutions of (9.1), where no subsum vanishes, by $W^*(d) := A(d, 1)$. Since all nontrivial subsums are of the same shape and there are at most 2^d subsums, we get that there are at most

$$2^d e^{d^2} \left(\log(g+2)\right)^{d-2} (ed)^{(d-1)(d+1)(s+1)}$$

pairs (n, m) in this case. Altogether, we get at most

$$e^{d^2+(6d)^{3d}} \left(\log(g+2)\right)^{d-2} (ed)^{(d-1)(d+1)(s+1)}$$

solutions (n, m) of (7.1).

Now let $\gamma_1, \gamma_2 \in F^*$ be given. In exactly the same way as above, Lemma 6.3 implies that $\gamma_1 x_i, \gamma_2 x_j \in \mathbf{K}^*$ or $\gamma_1 x_i, \gamma_2 x_{d+j} \in \mathbf{K}^*$ for $1 \le i < j \le d$ gives rise to at most one pair (n, m). Because of the fact that $\alpha_1(x), \ldots, \alpha_d(x)$ are not in $\mathbf{K}^*$ (and, therefore, also $\alpha_1(P(x)), \ldots, \alpha_d(P(x))$ are not in $\mathbf{K}^*$) it follows that $\gamma_1 x_{d+i}, \gamma_2 x_{d+j} \in \mathbf{K}^*$ $(1 \le i < j \le d)$ induces at most one m and we can use the arguments from above to get an upper bound for the number of (n, m) with this property.

From this it follows that we may take

$$k = e^{d^2+(6d)^{3d}} (\log(g+2))^{d-2} (ed)^{(d-1)(d+1)(s+1)}.$$

Now Proposition 3.5 implies that equation (7.1) has at most

$$\begin{aligned} &W(2d-1)\\ &\quad = e^{(d^2+(6d)^{3d})(2d-1)} e^{(2d-1)^2} (\log(g+2))^{(d-2)(2d-1)+2d-3} (2ed)^{(2d-2)2d(s+1)}\\ &\qquad \cdot (2ed)^{(d-1)(d+1)(2d-1)(s+1)}\\ &\quad \le e^{(d^2+2d-1+(6d)^{3d})(2d-1)} (\log(g+2))^{2d^2} (2ed)^{5d^3(s+1)} \end{aligned}$$

solutions (n,m), where no subsum vanishes.

Observe that exactly the same arguments work for the more general equation (7.3) (because we just use the structure of the solutions and the fact that we want to get solutions of $G_n(x) = c\,G_m(P(x))$) and that the bounds $W(2d-3), W^*(d)$ that we get can trivially be estimated from above by $W(2d-1)$.

By using the bound for the genus g of $F/\mathbf{K}$ (Lemma 6.1), we get

$$(\log(g+2))^{2d^2} \le \left(\log\left(d^{2d^2} \deg D(\deg P + 1)\right)\right)^{2d^2}.$$

Combining this bound with the upper bounds calculated in Sections 7 and 8, which add up to

$$d \cdot \left(W(2d-1) + 2^d \max\left\{W^*(d), e^{(6d)^{3d}}\right\} e^{(6(d-1))^{3(d-1)}} + 4^d W(2d-3)\right) + d!,$$

and using the bound for the cardinality of S (Lemma 6.2), we get the following bound:

$$C(d, A_0, D, P) = e^{(6d)^{4d}} \left(\log\left(d^{2d^2} \deg D(\deg P+1)\right)\right)^{2d^2} (2ed)^{30d^3 d!^2 \deg A_0 \deg P}$$

for the number of pairs (n,m) of integers with $n \neq m$ such that $G_n(x) = c\,G_m(P(x))$.

10. Calculation of the bounds for Theorems 2.5 and 2.7

We first have to show that (7.1), which we have shortened to

$$\beta_1 x_1 + \cdots + \beta_d x_d + \beta_{d+1} x_{d+1} + \cdots + \beta_{2d-1} x_{2d-1} = 1,$$

where $x_1, \ldots, x_{2d-1} \in \Gamma \subset U_S$ has at most $W(2d-1)$ nondegenerate solutions (n,m), i.e., solutions where no nontrivial subsum of the left-hand side of (7.1) vanishes. This follows from the Main Theorem on S-unit equations over fields of characteristic zero due to Evertse, Schlickewei and Schmidt (Theorem 3.1). But instead of applying it directly to the group Γ^n, which would yield $W(2d-1) = e^{(6(2d-1))^{3(2d-1)}(2d+1)}$, we apply it to the subgroup $\tilde{\Gamma}$ of $(F^*)^{2d-1}$ generated by

$$(\alpha_1(x), \ldots, \alpha_d(x), 1, \ldots, 1)$$

and

$$\left(\alpha_d(P(x))^{-1}, \ldots, \alpha_d(P(x))^{-1}, \frac{\alpha_1(P(x))}{\alpha_d(P(x))}, \ldots, \frac{\alpha_{d-1}(P(x))}{\alpha_d(P(x))}\right).$$

Thus the rank of $\tilde{\Gamma}$ is at most 2. Therefore, we get

$$W(2d-1) = e^{(6(2d-1))^{3(2d-1)} \cdot 3}$$

for the number of nondegenerate solutions of (7.1). In the same way we also obtain for the more general equations (7.3)

$$W(n) = e^{(6n)^{3n}\cdot 3}$$

and for $W^*(d)$ we can actually use Theorem 3.2 to get

$$W^*(d) = e^{(6d)^{3d}}.$$

Combining these bounds with the upper bounds calculated in Sections 7 and 8 yields

$$d \cdot \left(W(2d-1) + 2^d \max\left\{W^*(d), e^{(6d)^{3d}}\right\} e^{(6(d-1))^{3(d-1)}} + 4^d W(2d-3)\right) + d!$$

and, therefore, we get the bound

$$C(d) = e^{(12d)^{6d}}$$

for the number of pairs (n, m) of integers with $n \neq m$ such that $G_n(x) = G_m(P(x))$.

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